The Topological Structure of the QCD Vacuum

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Nobel prize in physics (2008)

"for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"

Yoichiro Nambu

Nobel prize (1/2)
Physics 2008

Chiral symmetry is one of the most subtle features of massless Dirac fermion, as a consequence of relativity + QM. Chiral symmetry leads to the conservation of the chiral charge (handedness).

If the chiral symmetry is spontaneously broken, then there exists massless Goldstone bosons (the number of GBs is equal to the number of broken generators).

Now, if the fermion has a small mass m (e.g., from EW breaking), then the mass of the Goldstone boson behaves as

$$m_{PS}{}^2 = \frac{4\Sigma}{f_{PS}{}^2} m$$

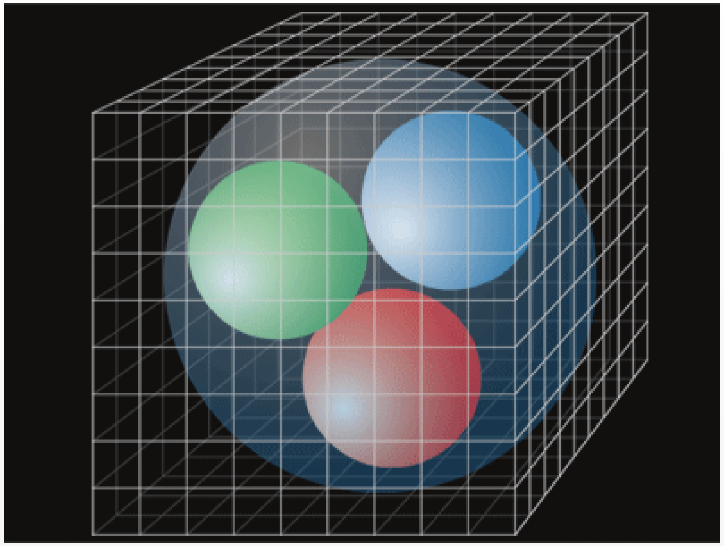
$$\Sigma \equiv \langle \bar{\psi}\psi \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \frac{1}{V} \mathrm{Tr}[(D+m)^{-1}]$$

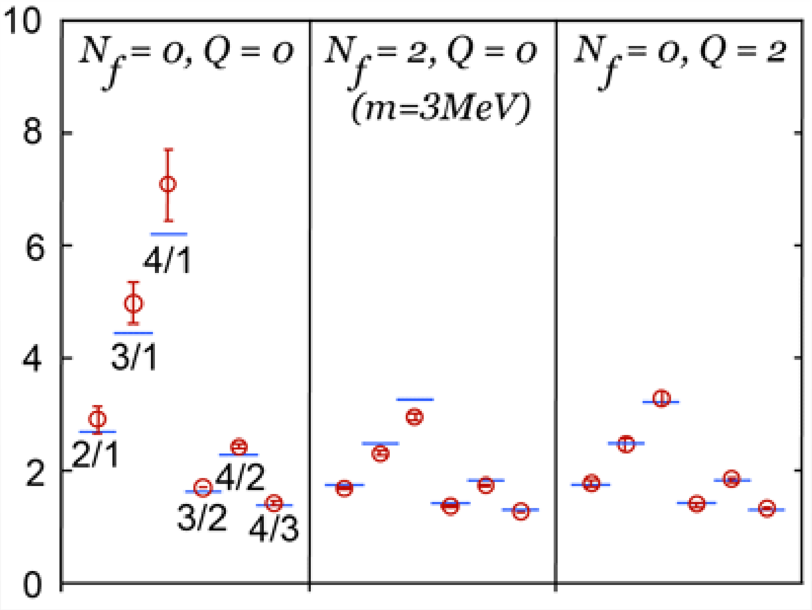
$\Sigma \neq 0 \Leftrightarrow$ chiral symmetry is spontaneously broken

A longstanding problem in particle physics is to show that the chiral sym. in QCD is spontaneously broken.

Spontaneous chiral symmetry breaking in QCD was reproduced on supercomputer IBM BlueGene/L (57 Tflops peak performance) at KEK, Japan.

[JLQCD and TWQCD Collaborations, Phys. Rev. Lett.98(2007)172001; Phys. Rev. D76 (2007) 054503, Phys. Rev. Lett. 101(2008) 202004]





Eigenvalues of quarks: QCD in a small box has discrete eigenvalues of quarks. This plot shows their ratios. "2/1", "3/1" etc. denote the ratio of second-to-first or third-to-first eigenvalues.

$$\Sigma^{\overline{MS}}(2\ \text{GeV}) = (251 \pm 7 \pm 11\ \text{MeV})^3$$

[JLQCD and TWQCD Collaborations, Phys. Rev. Lett.98(2007)172001; Phys. Rev. D76 (2007) 054503, Phys. Rev. Lett. 101(2008) 202004]

OUTLINES

- Introduction
- Topological susceptibility
- Topological structure of QCD vacuum
- Outlooks

The Challenge of QCD

At the hadronic scale, $g(r) \simeq 1$, perturbation theory is incapable to extract any quantities from **QCD**, nor to tackle the most interesting physics, namely, the **spontaneously chiral symmetry breaking** and the **color confinement**

To extract any physical quantities from the first principles of **QCD**, one has to solve **QCD** nonperturbatively.

A viable nonperturbative formulation of **QCD** was first proposed by **K. G. Wilson** in 1974.

But, the problem of lattice fermion, and to **formulate exact chiral symmetry on the lattice had not been resolved until 1992-98**.

Basic notions of Lattice QCD

1. Perform Wick rotation: $t \to -ix_4$, then $\exp(iS) \to \exp(-S_E)$, and the expectation value of any observable O

$$\langle O \rangle = \frac{1}{Z} \int [dA][d\psi][d\bar{\psi}] O(A,\psi,\bar{\psi}) e^{-S_E}$$

$$Z = \int [dA][d\psi][d\bar{\psi}] e^{-S_E}$$

(Recall that the divergences in QFT, which requires reg. and ren. , stemming from ∞ d.o.f. , and proximity of any field operator)

2. **Discretize the space-time as a 4-d lattice** $L^4 = (Na)^4$ with lattice spacing a. Then the path integral in QFT becomes a well-defined multiple integral which can be evaluated via Monte Carlo

$$\langle O \rangle = \frac{1}{Z} \int \prod_i dA_i \prod_j d\psi_j \prod_k d\bar{\psi}_k O(A,\psi,\bar{\psi}) e^{-S_E}$$

Gluon fields on the Lattice

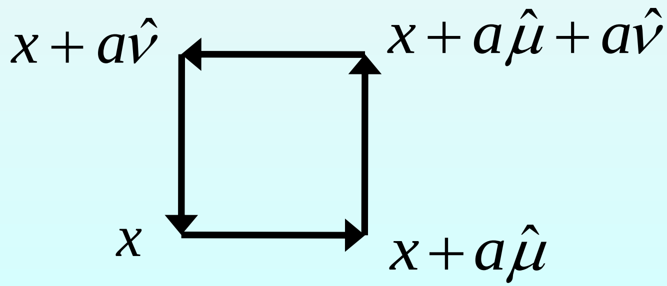
The $SU(3)$ color gluon field $A_\mu(x)$ are defined on each link connecting x and $x + a\hat{\mu}$, through the link variable

$$U_\mu(x) = \exp\left[iagA_\mu\left(x + \frac{a}{2}\hat{\mu} \right) \right]$$

Then the gluon action on the lattice can be written as

$$S_g[U] = \frac{6}{g^2} \sum_{\text{plaquette}} \left[1 - \frac{1}{3} \mathrm{Re}\ tr\left(U_p \right) \right] \xrightarrow{a \to 0} \int d^4x \frac{1}{2} tr\left[F_{\mu\nu}(x) F_{\mu\nu}(x) \right]$$

where $U_p = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)$



Lattice QCD

The QCD action $S = S_G(U) + \bar{\psi} D(U) \psi$

where $S_G(U)$ is the action of the gluon fields

$$\bar{\psi} D(U) \psi \equiv \bar{\psi}^f_{a\alpha x} D_f(U)_{a\alpha x, b\beta y} \psi^f_{b\beta y}$$

$f = u, d, s, c, b, t$	flavor index
$a, b = 1, 2, 3$	color index
$\alpha, \beta = 1,2,3,4$	Dirac index
$x, y = 1, \cdots, N_{\text{sites}} = N_x N_y N_z N_t$	site index

For example, on the $16^3 \times 32$ lattice, D is a complex matrix of size $1{,}572{,}864 \times 1{,}572{,}864$

$$\langle \mathcal{O}(\bar{\psi}, \psi, U) \rangle = \frac{\int dU d\bar{\psi} d\psi \, \mathcal{O}(\bar{\psi}, \psi, U) e^{-S}}{\int dU d\bar{\psi} d\psi \, e^{-S}} = \frac{\int dU \, \Theta(D^{-1}, U) \det(D) e^{-S_G}}{\int dU \det(D) e^{-S_G}}$$

The Challenge of Lattice QCD

- To have lattice volume large enough such that $m_\pi L \gg 1$.

  So far, the lightest u/d quark cannot be put on the lattice.

  To use ChPT to extrapolate lattice results to physical ones.

- To have lattice spacing small enough such that $m_q a \ll 1$.

- To meet the above two conditions:

  The lattice size should be at least $100^3 \times 200$.

  The computing power should be around Petaflops.

Current Status / The State of the Art

- RBC and UKQCD Collaborations (20-25 members)
  with ~ (10+10) Tflops (peak)
  ~(2+2) Tflops (sustained)
  lattice fermion: domain-wall fermion
  largest lattice: $32^3 \times 64$

- JLQCD and TWQCD Collaborations (10-15 members)
  with ~ (57+4) Tflops (peak)
  (8+2) Tflops (sustained)
  lattice fermion: overlap fermion
  largest lattice: $24^3 \times 48$

The QCD Vacuum

- The vacuum (ground state) of any nontrivial QFT constitutes various quantum fluctuations.
- These quantum fluctuations are the origin of many interesting and important nonperturbative physics.
- In QCD, each gauge configuration possesses a well-defined topological charge Q with integer value.

$$Z_Q = \int [dA][d\psi][d\bar{\psi}] e^{-S_E[Q]}, \quad Z(\theta) = \sum_{Q=\cdots,-1,0,+1,\cdots} e^{-i\theta Q} Z_Q$$

- Thus it is important to determine the topological charge fluctuation in the QCD vacuum.

Topological susceptibility is defined as

$$\chi_t = \int d^4x \langle \rho(x) \rho(0) \rangle$$

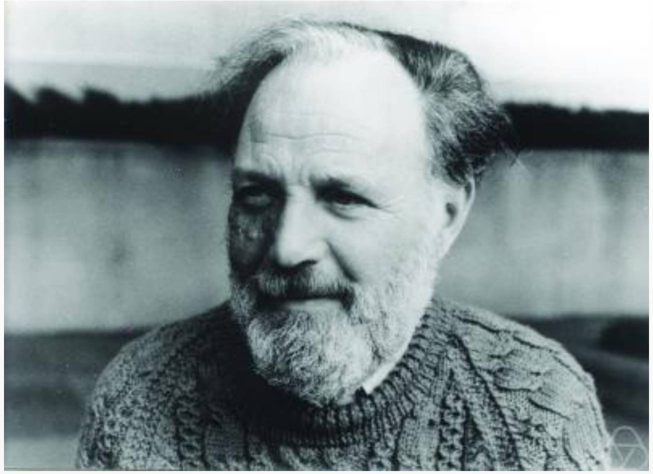
where $$\rho(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\lambda\sigma} \mathrm{tr}\left[F_{\mu\nu}(x) F_{\lambda\sigma}(x) \right]$$

$$Q_t \equiv \int d^4x \rho(x) = \text{ integer}$$

Thus $$\chi_t = \int d^4x \langle \rho(x) \rho(0) \rangle = \frac{\langle Q_t^2 \rangle}{V}$$

However, on a lattice, it is difficult to extract $\rho(x)$ unambiguously from the link variables !

Michael Atiyah, Abel prize(2004)
Fields Medal(1966)



Isadore Singer, Abel Prize(2004)

We turn to the Atiyah-Singer index theorem

$$Q_t \equiv \int d^4x \rho(x) = \mathrm{index}(D) = n_+ - n_-$$

where $n_\pm$ is the number of zero modes with $\pm$ chirality of the massless Dirac operator $D = \gamma^\mu(\partial_\mu + igA_\mu)$.

Since $m\int d^4x\,\mathrm{tr}\left[\gamma_5\left(D+m\right)^{-1}_{x,x}\right]=n_+-n_-=\int d^4x\,\rho(x)$

we can use $\rho'(x)\equiv m\ \mathrm{tr}\left[\gamma_5\left(D+m\right)^{-1}_{x,x}\right]$

to measure the topological susceptibility

In order to compute $\rho'(x)$ at any x, we need to compute the eigenmodes of D.

In practice, we only need to project about $50\sim100$

low-lying eigenmodes of $D=m_0\left(1+\gamma_5\frac{H}{\sqrt{H^2}}\right)$

Some important relations

Veneziano-Witten relation $\chi_t(\text{quenched}) = \frac{f_\pi^2 m_{\eta'}^2}{4N_f}$

Leutwyler-Smilga relation

$$\chi_t = \frac{\Sigma}{\left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s}\right)} + O(m_u^2), \quad N_f = 2+1$$

Banks-Casher relation

$$\Sigma = \pi\omega(0)$$

$\omega(0)$ is the density of near-zero modes of the massless Dirac operator $D = \gamma^\mu(\partial_\mu + igA_\mu)$

Topological susceptibility in 2+1 flavors QCD with domain-wall fermions

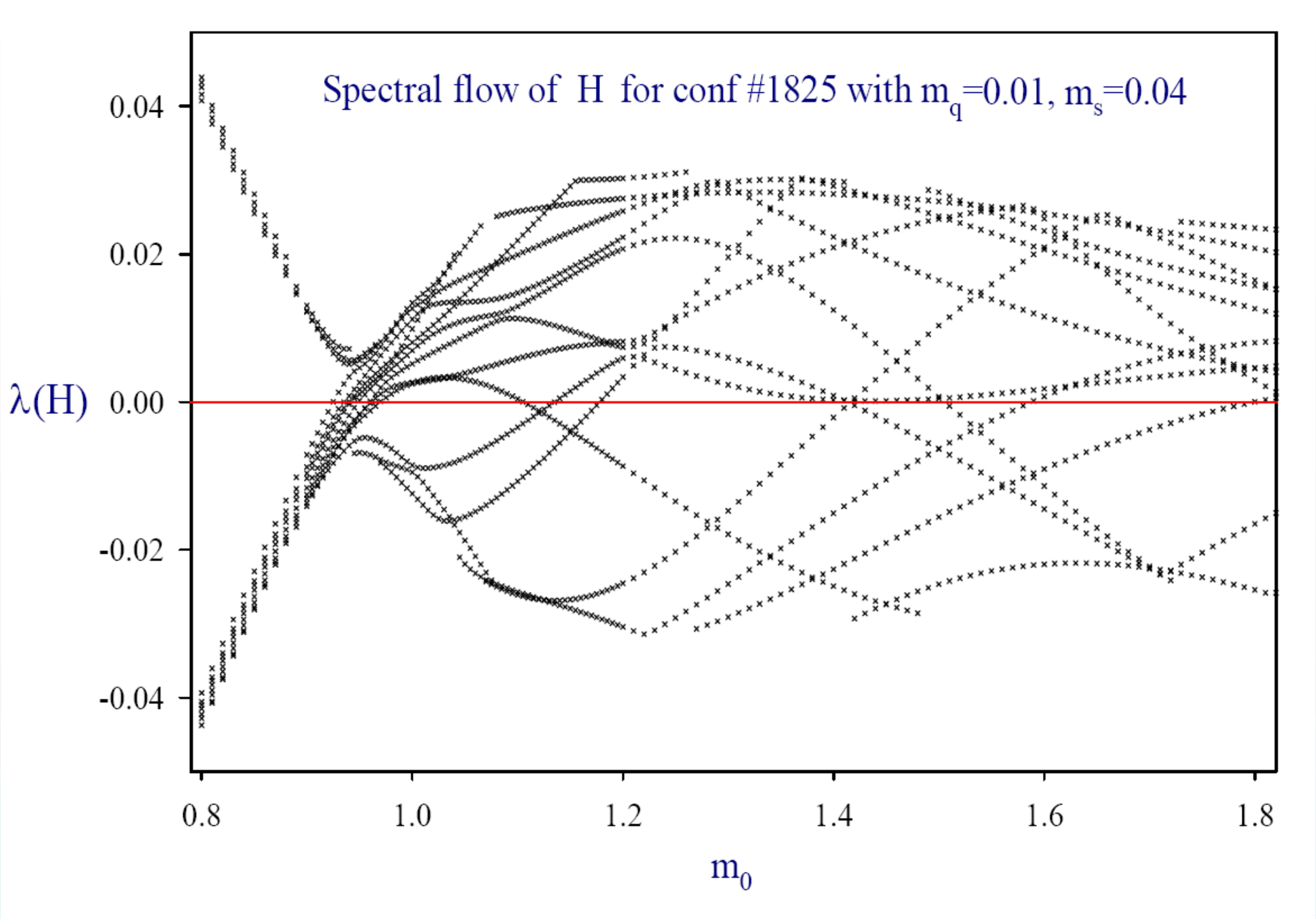
[TWQCD, arXiv:0810.3406, Phys. Lett. B (2008) in press]

- 2+1 flavors of DWF on the $16^3 \times 32$ lattice (L ≃ 2 fm) with $N_s = 16$, and $a^{-1} = 1.62(4)$ GeV
- $m_s = 0.04, \ \ m_u = 0.01, \ 0.02, \ 0.03,$ total 800+809+717 conf.
- Q of each conf. is evaluated using the spectral flow method

$$D = \frac{1}{2r}\left(1 + \gamma_5 \frac{H(-m_0)}{\sqrt{H^2(-m_0)}}\right), \quad \text{effective 4-dim Dirac operator}$$

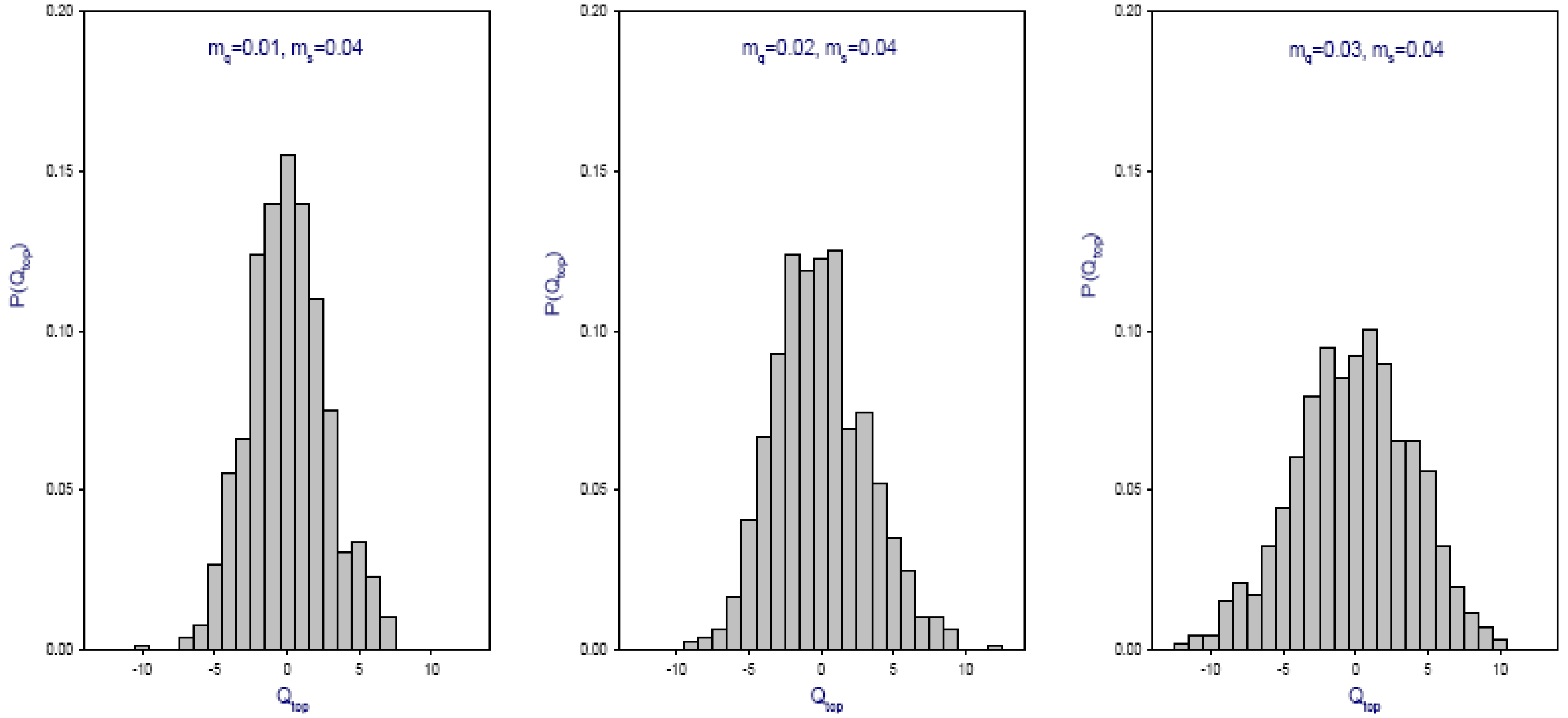
$$\text{index}(D) = n_+ - n_- = \text{Tr}[\gamma_5(1 - rD)]$$

$$= -\frac{1}{2}\text{Tr}\left(\frac{H}{\sqrt{H^2}}\right) = \frac{1}{2}(h_- - h_+)$$



The spectral flow of 12 lowest-lying eigenvalues of H. There are 10 net crossings from negative to positive, so the index = -10.

Histogram of topological charge distribution



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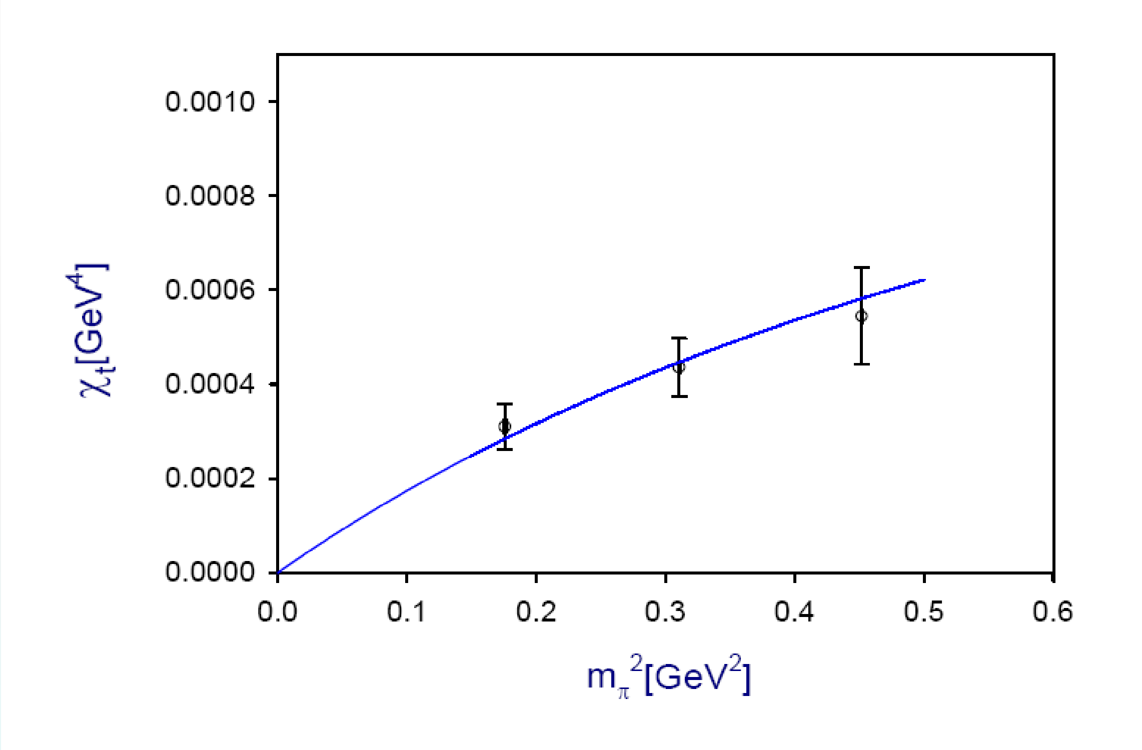


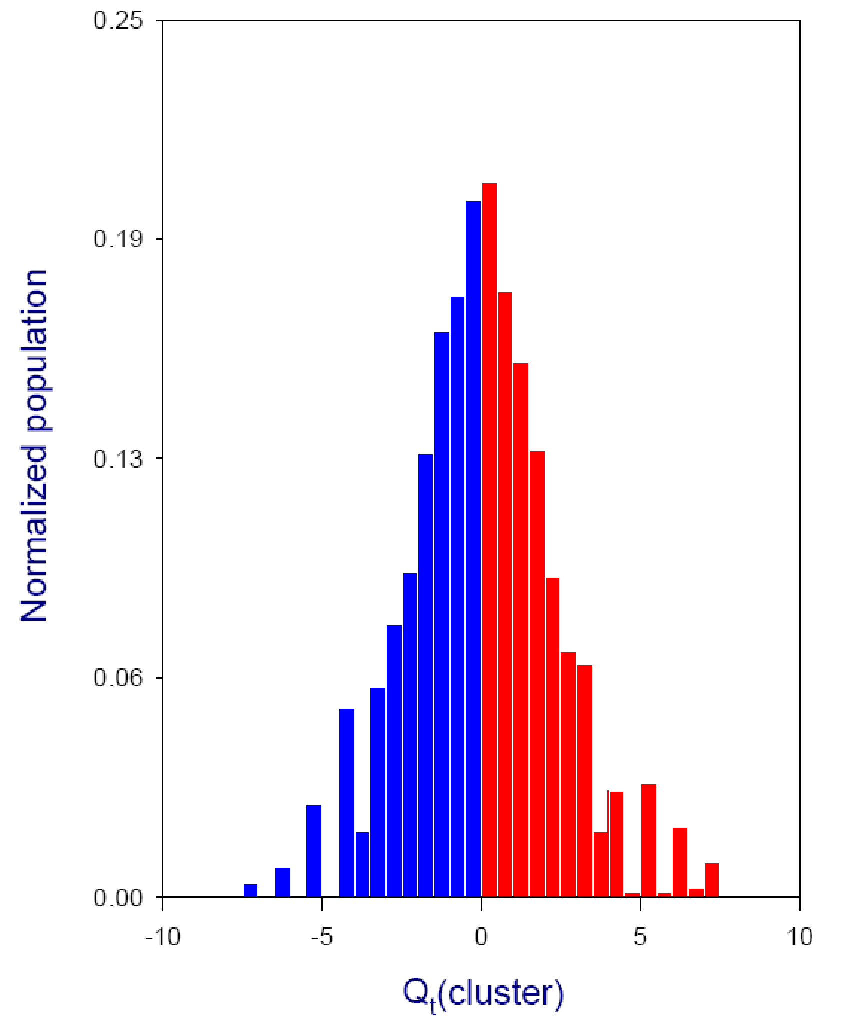
ChPT $\chi_t = \dfrac{\Sigma}{\left(\dfrac{1}{m_u} + \dfrac{1}{m_d} + \dfrac{1}{m_s}\right)} \Rightarrow \Sigma^{\overline{MS}}(2\text{ GeV}) = [259(6)(9)\text{ MeV}]^3$

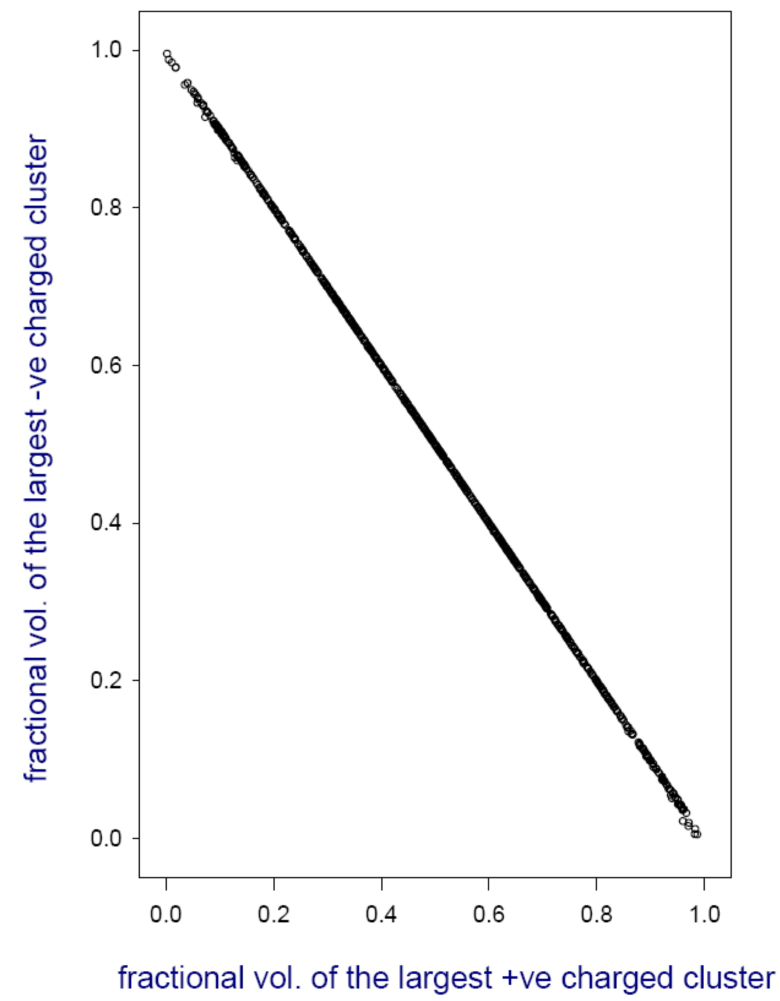
Topological structure of the QCD vacuum

[TWQCD Collaboration, in preparation]

- 2+1 flavors of DWF on the $16^3 \times 32$ lattice ($L \simeq 2$ fm) with $N_s = 16$, and $a^{-1} = 1.62(4)$ GeV
- $m_s = 0.04$, $m_u = 0.01$, 0.02, 0.03, total 800+809+717 conf.
- For each conf., project 50 pairs of low-lying eigenmodes
- Compute topological charge density $\rho(x) = m \, \mathrm{tr}\left[\gamma_5 \left(D + m \right)^{-1}_{x,x} \right]$ with low-lying eigenmodes.
- Identify topological clusters with positive or negative charge, by grouping any two neighboring sites with the same sign of $\rho(x)$

$N_f=2+1$, $m_u=0.01$, $m_s=0.04$, nev=50, 800 confs.





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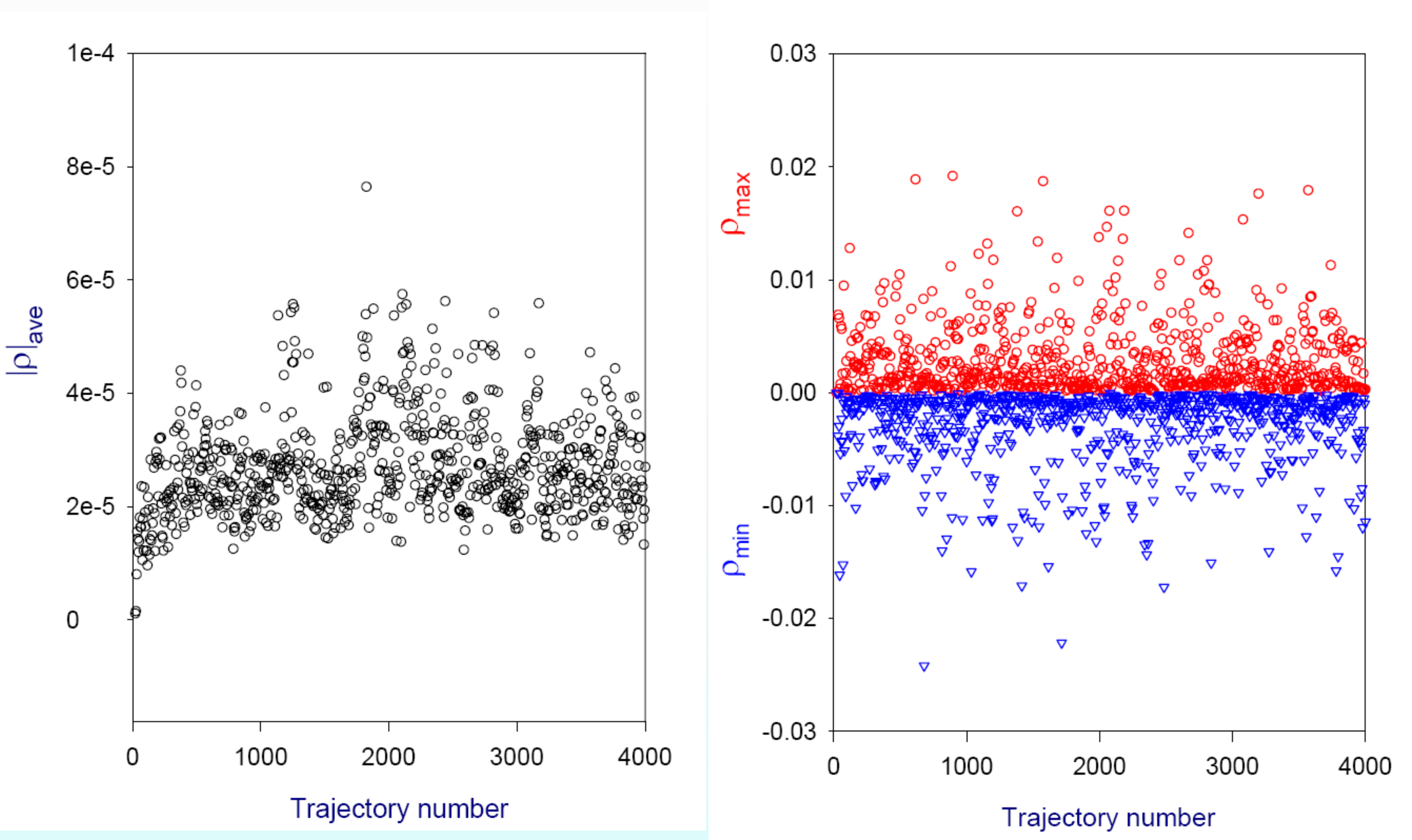
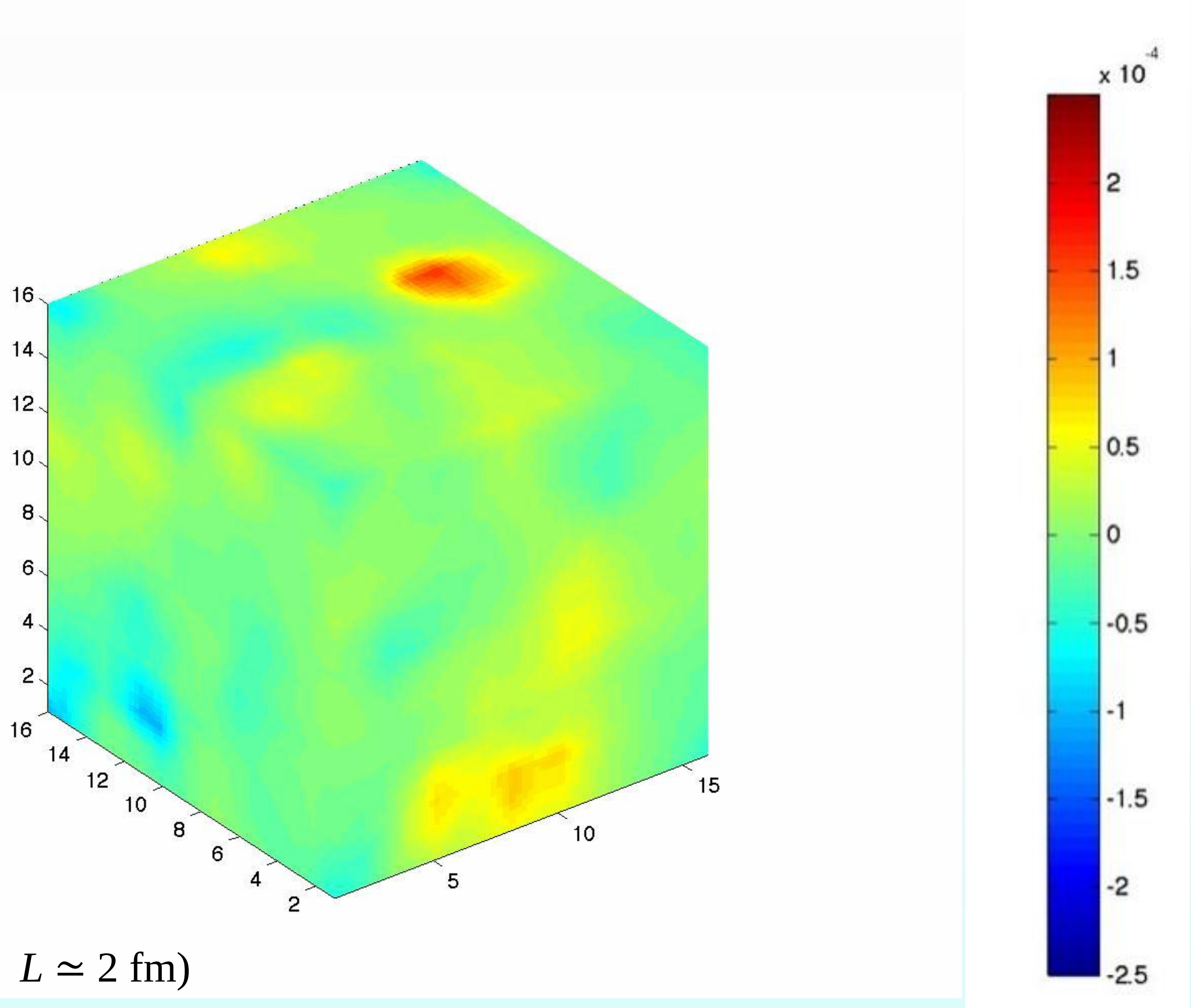


Quantum fluctuations in the QCD vacuum

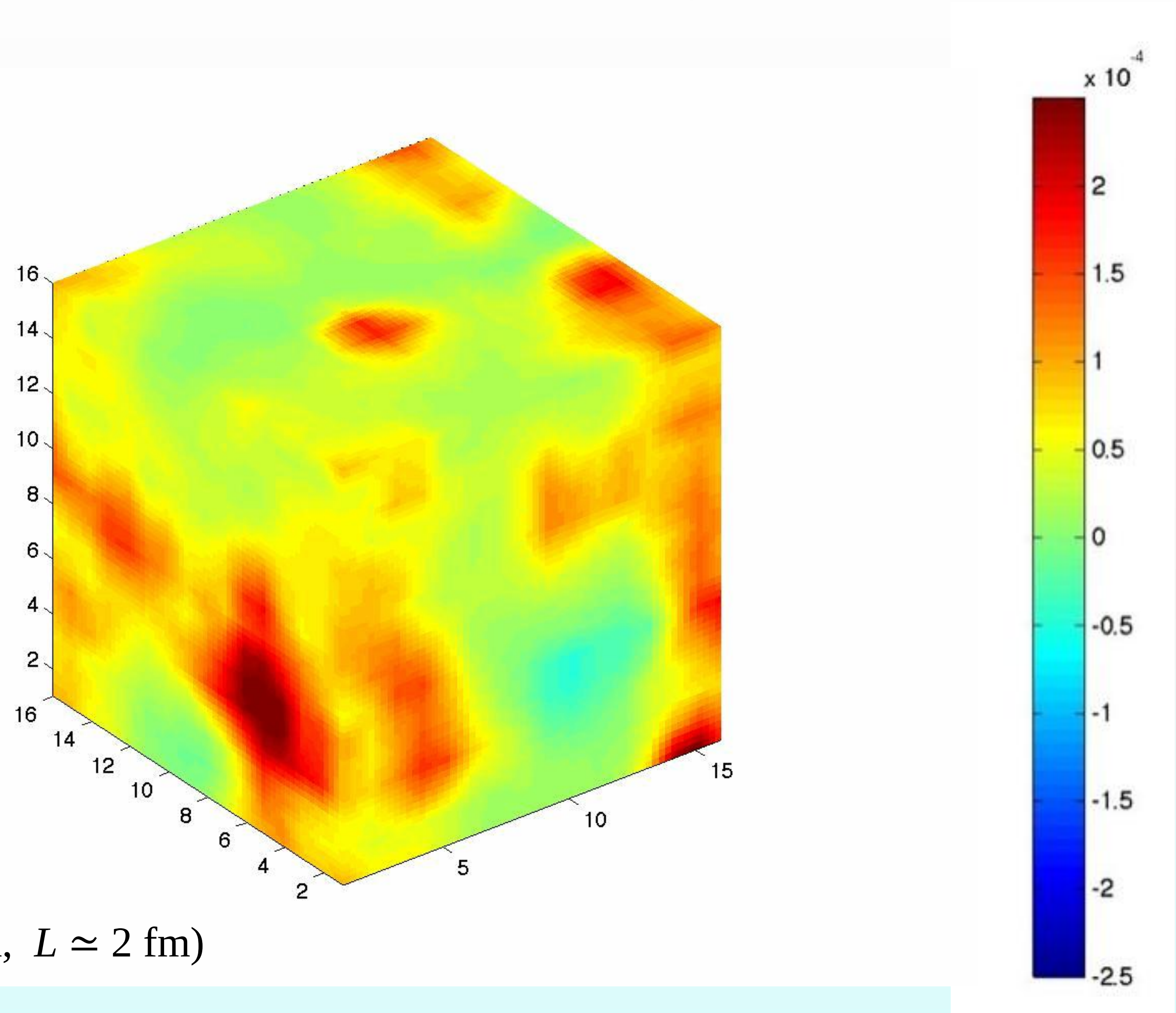
$Q_t = -2$



($a \simeq 0.12$ fm, $L \simeq 2$ fm)

Quantum fluctuations in the QCD vacuum

$Q_t = 7$



($a \simeq 0.12$ fm, $L \simeq 2$ fm)

Outlook

- To clarify the nature of QCD vacuum, whether it is more instanon-like, or more complicated 2-dim or 3-dim sheet-like structure. Namely, to test the (anti-)self-duality,

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}$$

- To identify the QCD vacuum fluctuations which are the most relevant to the mechanism of color confinement.